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ONE-DIMENSIONAL TRANSPORT FROM A HIGHLY CONCENTRATED, TRANSFER TYPE SOURCE

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Abstract—In both heat and mass transfer, situations arise in which an entity considered as a source/sink has strength which can only be expressed in terms of an unknown rate of source—flow field transfer. This occurs when transfer between the source and medium is driven by a dependent variable difference which is unknown, because the responding medium value is unknown. Manifold mathematical complexities arise when in addition the source is highly concentrated spatially, relative to the size of the overall domain. A 1-dim. convective–diffusive transport equation suitable for this case may be solved by simultaneous use of the Fourier transform and its inverse in the same equation, together with other transformation and manipulation. From the solution obtained for the case of constant source intensity, one may construct a general expression for the solution when source intensity varies arbitrarily in time. Explicit expressions are obtained for solution of the fundamental case of temporally sinusoidal source intensity.

NOMENCLATURE*

A,	amplitude, equation (36);
a,	see equation (B.1);
B,	see equation (A.8);
b,	complex constant, equation (40);
C,	concentration;
C_0,	initial concentration;
C_{ref},	reference concentration;
C_1,	see equation (B.1);
c,	see equation (B.1);
D,	diffusion coefficient;
F,	any appropriate function, equations (17);
f,	see equation (A.3);
H,	θ solution for unit value of θ_{sc};
h,	see equation (A.2);
i,	$\sqrt{(-1)}$;
k_i,	set of complex constants, see Appendix B;
L_τ,	Laplace transform with respect to τ;
Q,	source strength;
q,	non-dimensional source strength, equation (7);
Re,	real part of;
S,	source transfer coefficient;
s,	Laplace transform variable;
t,	time;
V,	flow rate;
x,	position.

Greek symbols

β,	Fourier transform variable, equation (17);
δ,	Dirac delta function;
ε,	one-half the length of a steady-state control volume, see Fig. 2;
η,	non-dimensional space variable, equation (10a);
θ,	dimensionless dependent variables;
θ_{sc},	θ value of source (source intensity);
θ_∞,	steady-state value of θ;
Π,	Laplace transform of $\bar{\phi}$;
τ,	nondimensional time, equation (10b);
ϕ,	see equation (14);
ϕ_i,	see equation (A.21);
$\bar{\phi}$,	Fourier transform of Φ;
$\bar{\phi}_i$,	see equation (A.13);
ω,	circular frequency, equation (36).

* Dummy variables of integration are written in the text as primed quantities. In general, they correspond to non-primed, non-dummy quantities represented by the same letter.

1. INTRODUCTION

A GENERAL 1-dim. transport equation, applicable when sources and sinks are present is

$$\frac{\partial \theta}{\partial t} + V\frac{\partial \theta}{\partial x} - D\frac{\partial^2 \theta}{\partial x^2} = Q. \quad (1)$$

Here V and D are considered to be specified constants, and Q is the source strength. The dependent variable θ can represent temperature, concentration of a solute, or any other quantity for which such a transport equation is appropriate.

A great deal of effort has been devoted to the solution of this equation and its cousins under various initial and boundary conditions. Source effects may be of particular interest or difficulty, whether arising from phase change, deposition [1], reaction [2], adsorption [3], external lateral transfer [4], radioactive decay [5] or some other mechanism. Whether source type activity may be represented logically through boundary conditions or not depends on the particular case. Frequently a combination of source/sink effects occurs, as in the cases of superposed equilibrium and kinetic adsorption [3], point injection of particulates which also precipitate throughout the domain [6], or transport of a chemical through a sorbing medium in which there is also lateral, intra-aggregate diffusion [7].

In some practical situations, the strength of the source is proportional to the difference between the value of θ in the flowing medium and the value of θ associated with the source. That is

$$Q = S(\theta_{sc} - \theta), \qquad (2)$$

where θ_{sc} is the source intensity.

A relation such as equation (2) might apply for example to solute transfer from a deposit in a stream or fluid-bearing porous medium. If θ corresponds to temperature, then S is proportional to the heat transfer coefficient between the medium and the source, and $(\theta_{sc} - \theta)$ is the temperature difference which drives exchange with the source.

Schneider [8], for example, uses similar representation of Q to describe heat transfer to the solidified layer on a cold wall past which liquid flows. In his analysis, growth of the solidified layer is transient, but the first and third terms in equation (1) are neglected. Crank [9] reviews many applications of such a source representation in (1), with no convection, and Mikhailov and Özisik [10] develop a generalized method of analysis which they apply to some of the same problems.

In situations where the flow field is macroscopically uniform, use of a simple 1-dim. description such as equations (1) and (2) may obviously be warranted. While the details of flow through a sorbing or reacting porous medium are extremely complex and multidimensional at the microscopic level, macroscopically the flow field may be uniform and the matrix–fluid transfer may well be summed up in the form (2). Such a simple formulation may also be desirable if one is to avoid quite formidable flow and transfer problems within larger, constructed devices, as in the case of transfer to or from a bank of tubes in a cross flow. Use of a system such as (1), (2) gains additional cogency if the flow field is much larger in the longitudinal than transverse directions. The system also appeals when the source is quite concentrated spatially relative to the overall domain (illustrated in Fig. 1) so that its internals are not to be analyzed, while one focuses more on the effects of its transfer with the larger domain.

Equations along the lines of (1), (2) may make sense in connection with complex internal flow transfer where one wishes to retain diffusive, convective and lateral influx mechanisms, and some geometrical simplicity cooperates in a rational system of one-dimensionalization. We may consider, for example, the problem of Faghri and Welty [11], entailing axial flow in a uniform pipe, with conduction, convection and circumferentially varying surface heat flux. With axial, circumferential, and radial variation, the physical problem is 3-dimensional. In many practical situations, it is not feasible to observe or analyze relevant quantities in such detail, while cross-sectionally integrated velocity, temperature, or lateral heat flux are more tractable. Özisik and Mulligan [12], for example, consider a physically related problem in which freezing occurs on the pipe wall. In addition to simplifying matters by neglecting the conduction term, they retain radial conduction but assume slug flow in the liquid. This facilitates a Hankel transform of the governing equation with respect to the radial coordinate, which leaves a system like (1), (2) with D and θ_{sc} equal to zero. Considering flow of a solute in a tube, Aris [13] succeeds in casting the problem into 1-dimension, using cross-sectional average velocity and concentration, with a rational accounting for radial variation and interactions through a specific addition to the diffusion coefficient. Usually when averaging procedures are applied to obtain equations like (1), dispersion type terms are produced at the scale of the averaged quantities which are more difficult to specify *a priori* (e.g. see [14, 15]). Thus one of the uses of analytical solutions such as that below is to provide a means of inferring unknown parameter values from large scale behavior of a dependent variable. Cross-sectional averaging of the governing equations used by Faghri and Welty [11], if done appropriately, would produce an equation such as (1) with detailed multi-dimensional interactions buried in Q (or S).

A moving heat source may also produce equations along the lines of (1), (2) if one proceeds in a coordinate system attached to the source itself. Ling and Yang [16] solve for the temperature distribution in a semi-

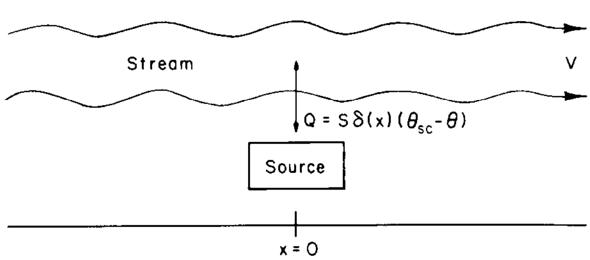

FIG. 1. Schematic representation of transfer between a highly concentrated source and a flow field.

infinite solid, in contact with a moving, spatially concentrated heat source. In the frame attached to the source, both convection and conduction appear in the solid. Their equations pertain to a steady-state in the moving frame, and correspond to the set above for a very thin solid, with the incident flux expressed as in (2), with S non-zero only over a limited domain. Analyzing 1-dim. ablation of a semi-infinite solid, Landau [17] transforms to a coordinate system in which the phase change location is fixed, a technique found useful by many subsequent investigators (e.g. see Ferriss [18], and Lynch and O'Neill [19]). In Landau's method a convection term is added by the coordinate motion, so that an equation in the general form of (1) results. If the phase change is included as a source, then the source is entirely concentrated at the surface point. For convective transfer on the surface, (2) contains an S concentrated infinitely.

In this paper, transient, 1-dim. convective–diffusive transport from a highly concentrated source is treated. The source is considered to be negligible in extent, relative to the overall domain. When equation (2) is modified appropriately, and substituted into (1), one obtains the governing equation to be solved below:

$$\frac{\partial \theta}{\partial t} + V\frac{\partial \theta}{\partial x} - D\frac{\partial^2 \theta}{\partial x^2} + S\,\delta(x)(\theta - \theta_{sc}) = 0, \quad (3)$$

subject to

$$\lim_{x \to \pm \infty} \theta = 0, \qquad \theta(x, 0) = 0 \quad (4)$$

where $\delta(x)$ is the Dirac delta function. Figure 1 illustrates the situation schematically. We assume that θ may always be chosen such that the initial and boundary conditions may be expressed as in (4), and that θ is non-dimensional. If solute transport is contemplated, θ could be

$$\theta \equiv \frac{C - C_0}{C_{ref} - C_0} \quad (5)$$

where C_0 is a constant, initial concentration, and C_{ref} is some reference concentration, for example a concentration characteristic of the source. For heat transfer, temperature is simply substituted for concentration in (5).

In what follows, the system (3), (4) is solved analytically for both constant and time-dependent θ_{sc}. This is accomplished primarily by the application of transform techniques, for the solution of both differential and integral equations along the way. The essential mathematical problem in this approach is caused by the presence of the delta function multiplied times θ. The reader obtains access to all concepts and procedures used to treat this problem, either by direct inclusion of detail in the text, in references, or in the appendices.

All solutions arrive in closed form, with transient terms containing error functions which can easily be evaluated. In addition to their predictive value when system parameters are known, these solutions provide a means of estimating those parameters when dependent variable data is at hand instead. Relations of the form (2) usually constitute idealizations of source—medium transfer in any practical situation. While one may retrieve some real world complexity through arbitrarily complex forms of S, we are interested here in an analytical solution, and will therefore assume a mathematically beneficent physical situation, in which the assumption of constant S, V and D is acceptable. The results obtained under this assumption furnish qualitative indications of system response, may frequently be valid at least as a first approximation, and may be sufficient for engineering and design purposes. Numerical procedures designed for more general situations of the same ilk may be tested against these results, and it is hoped that the mathematical methods and manipulations enlisted may prove useful in attacking related problems.

2. THE STEADY STATE

Figure 2 shows schematically a steady state, approached at some time sufficiently long after the initial conditions applied. One may obtain a steady-state value of θ by integrating over a sufficiently large control volume containing the source, and thus balancing inflow, outflow, and source infusion of θ. As used here and in what follows, the term "steady-state" means a local steady state, that is, an asymptotically approached constant distribution of θ about any finite point in space. Of course, if one proceeds further downstream, eventually he will encounter a zone in which θ is still changing, and beyond that a zone where the source influence is negligible. The volume shown in Fig. 2 extends upstream beyond significant source influence, but not so far downstream so that the zone of active change is reached. If one assumes that the steady state exists as depicted, then the steady-state (maximum) value of θ is

$$\theta_{\infty} = \frac{q\theta_{sc}}{q+1} \quad (6)$$

where

$$q \equiv \frac{S}{V}. \quad (7)$$

Thus q is essentially a non-dimensional source strength. When S is high relative to V, this effective source strength is high and

$$\left.\begin{matrix} q \gg 1 \\ \theta_{\infty} \approx \theta_{sc} \end{matrix}\right\}. \quad (8)$$

For a relatively low source transfer coefficient

$$\left.\begin{matrix} q \ll 1 \\ \theta_{\infty} \approx q\theta_{sc} \end{matrix}\right\}. \quad (9)$$

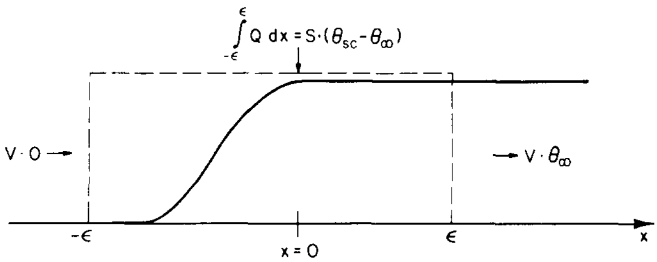

FIG. 2. Control volume, over which the governing equation is integrated to obtain steady-state information. It is assumed that the solution curve is sufficiently flat both up and downstream of the source so that diffusion across the volume boundaries is negligible.

Figure 3 shows $(\theta_\infty/\theta_{sc})$ as a function of q. The figure illustrates the information in equations (8) and (9), and shows how the zone of a maximum influence of q lies between values of about 0.1 and 10. Given θ_{sc} and an observed θ_∞ in some situation, one could evaluate q from the above expressions. From q one could also evaluate S or V, knowing one of them.

One may question the validity of θ_∞ as obtained above, both in general, and for any given point in space and time. More rigorous derivation of the steady-state expression (6) comes through the complete solution for θ over time and space, as presented next.

3. COMPLETE SOLUTION, CONSTANT θ_{sc}

Most coefficients can be eliminated from equation (3) by the following transformations:

$$\eta \equiv \frac{xV}{D}, \qquad (10a)$$

$$\tau \equiv \frac{V^2 t}{D}. \qquad (10b)$$

When the relation

$$\delta(x) = \frac{V}{D}\delta(\eta) \qquad (11)$$

[implied by the coordinate transformation (10a)], and the definition of q in (7) are used along with (10b), equation (3) becomes

$$\frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial\eta} - \frac{\partial^2\theta}{\partial\eta^2} + q\delta(\eta)(\theta-\theta_{sc}) = 0 \qquad (12)$$

subject to

$$\lim_{\eta\to\pm\infty} \theta = 0, \qquad \theta(\eta, 0) = 0. \qquad (13)$$

Linear systems such as (12), (13) are often amenable to treatment by integral transform techniques if the $\partial\theta/\partial\eta$ term is removed. This can be accomplished by the transformation [20, pp. 526–527]

$$\phi(\eta, \tau) \equiv \theta(\eta, \tau)\exp(-\eta/2 + \tau/4). \qquad (14)$$

Using (14) in (12), (13) one obtains

$$\frac{\partial\phi}{\partial\tau} - \frac{\partial^2\phi}{\partial\eta^2} + q\delta(\eta)[\phi - \theta_{sc}\exp(-\eta/2+\tau/4)] = 0, \qquad (15)$$

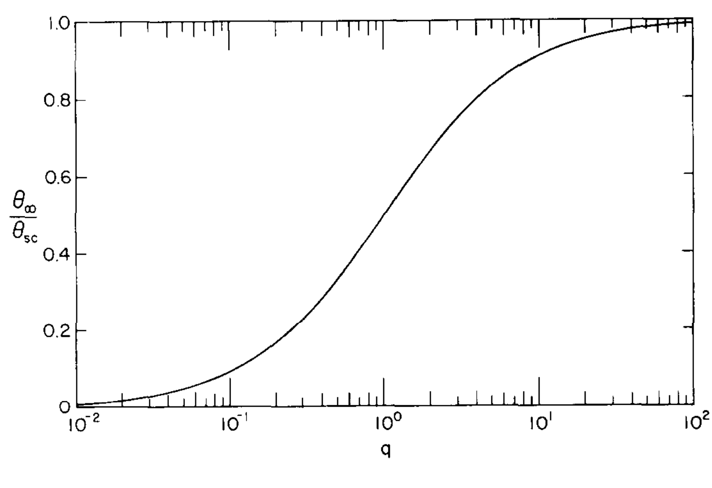

FIG. 3. Steady-state dependent variable value over source intensity, as a function of q. The range of greatest influence of q is evidently between approx. 0.1 and 10.

$$\lim_{\eta \to \pm\infty} \phi = 0, \tag{16a}$$

$$\phi(\eta, 0) = 0. \tag{16b}$$

One may now use the first part of the integral Fourier transform pair

$$\bar{F}(\beta) \equiv \int_{-\infty}^{\infty} \exp(i\beta\acute{\eta}) F(\acute{\eta}) \,d\acute{\eta} \tag{17a}$$

$$F(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\acute{\beta}\eta) \bar{F}(\acute{\beta}) d\acute{\beta} \tag{17b}$$

to reduce (15) to a differential equation in τ:

$$\frac{\partial \bar{\phi}}{\partial \tau} + \beta^2 \bar{\phi} + q\phi(0, \tau) - q\theta_{sc} \exp(\tau/4) = 0, \tag{18}$$

where $\bar{\phi}$ is the Fourier transform of ϕ,

$$\bar{\phi} = \bar{\phi}(\beta, \tau). \tag{19}$$

The boundary conditions (16a) at $\eta \to \pm\infty$ have been applied in the course of the transform integrations. Also, the condition (16a) was considered to express the fact that, for sufficiently large values of $|\eta|$, ϕ is essentially undisturbed relative to the initial condition, and hence its gradient is zero.

The initial condition (16b) on ϕ is transformed to

$$\bar{\phi}(\beta, 0) = 0. \tag{20}$$

The term $q\phi(0, \tau)$ appears because, by the nature of the delta function,

$$\int_{-\infty}^{\infty} \phi(\eta, \tau)\,\delta(\eta) \exp(i\beta\eta)\, d\eta = \phi(0, \tau). \tag{21}$$

The system (18), (20) may be solved for $\bar{\phi}$ by elementary methods, and the result is

$$\bar{\phi}(\beta, \tau) = -q \exp(-\beta^2 \tau) \int_0^{\tau} \exp(\beta^2 \acute{\tau})\, \phi(0, \acute{\tau})\, d\acute{\tau} + \frac{q\theta_{sc}}{1/4 + \beta^2} [\exp(\tau/4) - \exp(-\beta^2 \tau)]. \tag{22}$$

At this point the crux of the problem is apparent. Because the delta function operates against the dependent variable itself, application of any integral transform leaves an equation containing both untransformed $[\phi(0, \tau)]$ as well as transformed $[(\bar{\phi})]$ descendents. The cure for this problem lies in the reexpression of $\phi(0, \tau)$ in terms of $\bar{\phi}$, using (17b) as detailed below.

Equation (17b) implies

$$\phi(0, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}(\acute{\beta}, \tau) \exp(-i\acute{\beta}\eta)\, d\acute{\beta} \Bigg|_{\eta = 0} \tag{23}$$

that is,

$$\phi(0, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}(\acute{\beta}, \tau)\, d\acute{\beta}. \tag{24}$$

Introduction of (24) into (22) produces an equation in $\bar{\phi}$ only, containing two integrals:

$$\bar{\phi}(\beta,\tau) = -q \int_{\acute{\tau}=0}^{\tau} \exp[-\beta^2(\tau - \acute{\tau})] \frac{1}{2\pi} \int_{\acute{\beta}=-\infty}^{\infty} \bar{\phi}(\acute{\beta}, \acute{\tau})\, d\acute{\beta} d\acute{\tau} + \frac{q\theta_{sc}}{\frac{1}{4} + \beta^2} [\exp(\tau/4) - \exp(-\beta^2 \tau)]. \tag{25}$$

This equation may be solved for θ, via ϕ, via $\bar{\phi}$, via an additional transform and the ingenuities detailed in Appendix A. The result is

$$\theta(\eta, \tau) = \frac{q^2\theta_{sc} \exp[(q+1)\eta/2 + (q^2-1)\tau/4]}{1-q^2} \times \operatorname{erfc}[(q/2)\sqrt{\tau} + (\eta/2)/\sqrt{\tau}] + \frac{q\theta_{sc} \exp(\eta)}{2(q-1)} \operatorname{erfc}\left[\frac{1}{2}\sqrt{\tau} + (\eta/2)/\sqrt{\tau}\right] - \frac{q\theta_{sc}}{2(q+1)} \operatorname{erfc}\left[\frac{1}{2}\sqrt{\tau} - (\eta/2)/\sqrt{\tau}\right] + \frac{q\theta_{sc}}{q+1}. \tag{26}$$

Using a series expression for erfc(z) in the first two terms [21], one may easily see that they go to zero as η increases, at any given point in time. The last two terms combine in the form of

$$\frac{q\theta_{sc}}{q+1}\left[1 - \frac{1}{2}\operatorname{erfc}(z)\right].$$

As η increases, $z(\eta)$ goes to negative infinity, and this expression goes to zero as well. Thus the solution behaves in accordance with the assumed far-field condition. Examining the specific expressions involved in (26), one also sees that, as τ increases, ever greater values of η must be reached before a zero value for θ is approached. In other words, as time proceeds, the influence of the source is felt further downstream.

The last term on the right in (26) represents the steady-state value of θ, in the form prophesied. Denoting that term as in (6), one may rewrite (26) in a somewhat simplified form, in which θ_{sc} does not appear explicitly:

$$\left(\frac{\theta}{\theta_{s}}\right) = \left(\frac{q}{1-q}\right)\exp[(q+1)\eta/2+(q^2-1)\tau/4]$$
$$\times\ \mathrm{erfc}[(q/2)\sqrt{\tau}+(\eta/2)/\sqrt{\tau}]$$
$$+\ \frac{(q+1)\exp(\eta)}{2(q-1)}\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau} + (\eta/2)/\sqrt{\tau}\right]$$
$$-\ \frac{1}{2}\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau} - (\eta/2)/\sqrt{\tau}\right] + 1, \quad q \neq 1. \quad (27)$$

For all values of q and θ_{sc}, (θ/θ_{s}) ranges from zero initially to 1, as time increases.

For the very special case when $q = 1$, one may still obtain a solution from (27) by using l'Hôpital's rule, taking the limit as q approaches 1. The result is

$$\left(\frac{\theta}{\theta_{s}}\right)_{q=1} = \exp(\eta)\left\{\sqrt{\frac{\tau}{\pi}}\exp\{-[(1/2)\sqrt{\tau}+(\eta/2)/\sqrt{\tau}]^2\}\right.$$
$$\left. -\ \frac{1}{2}(1+\eta+\tau)\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}+(\eta/2)/\sqrt{\tau}\right]\right\}$$
$$-\ \frac{1}{2}\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}-(\eta/2)/\sqrt{\tau}\right]. \quad (27a)$$

The general result (26) was tested against a numerical solution of the system (12), (13), using Hermitian finite elements in space and finite differences in time [22]. Although there are slight oscillations in the numerical solution due to the steepness of the curve upstream of the source point (see Fig. 4), the very good agreement between the two solutions reinforces one's faith in the validity of (26).

4. THE SOURCE VICINITY, CONSTANT θ_{sc}

Equation (27) is readily simplified to express the solution in the vicinity of the source, that is, for $\eta = 0$:

$$(\theta/\theta_{s})_{\eta=0} = \frac{q\exp(q^2-1)\tau/4}{1-q}\mathrm{erfc}[(q/2)\sqrt{\tau}]$$
$$+\ \frac{1}{2}\left[\frac{q+1}{q-1} - 1\right]\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}\right] + 1. \quad (28)$$

This equation provides curves showing the rising time of (θ/θ_{s}) around the source, as a function of q.

5. TIME-DEPENDENT SOURCE INTENSITY

Equation (12) may be written

$$\frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial\eta} - \frac{\partial^2\theta}{\partial\eta^2} + q\delta(\eta)\theta = q\delta(\eta)\theta_{sc}. \quad (29)$$

In this form it is clear that the operator on the left is linear in θ, and solutions corresponding to various forcing functions on the right may be superposed. If at first $\theta_{sc}(\tau)$ is constant at $\theta_{sc}(0)$, then the corresponding solution will be

$$\theta = \theta_{sc}(0)\,H(\eta,\tau), \quad (30)$$

where $H(\eta,\tau)$ is the solution obtained from (26) for θ_{sc} held constant at unity. If at some subsequent time τ_1 the value of θ_{sc} is changed by $\Delta\theta_{sc}(\tau_1)$, then the corresponding solution will be

$$\theta = \theta_{sc}(0)\,H(\eta,\tau) + \Delta\theta_{sc}(\tau_1)\,H(\eta,\tau-\tau_1), \quad \tau > \tau_1. \quad (31)$$

In a similar manner one may construct a solution valid after a succession of n arbitrary step changes in θ_{sc}, by employing n concatenated superpositions:

$$\theta = \theta_{sc}(0)\,H(\eta,\tau) + \sum_{k=1}^{n}\Delta\theta_{sc}(\tau_k)\,H(\eta,\tau-\tau_k), \quad \tau > \tau_n. \quad (32)$$

Or, in the same spirit one may employ a convolution integral generalization of (32), suitable for continuous as well as step varying θ_{sc}:

$$\theta = \theta_{sc}(0)\,H(\eta,\tau) + \int_0^{\tau}\frac{\partial\theta_{sc}}{\partial\tau'}H(\eta,\tau-\tau')\mathrm{d}\tau'. \quad (33)$$

In some cases a change of variable may be more convenient:

$$\theta = \theta_{sc}(0)\,H(\eta,\tau) - \int_0^{\tau}\frac{\partial\theta_{sc}(\tau-\tau')}{\partial\tau'}H(\eta,\tau')\mathrm{d}\tau', \quad (34)$$

or, integrating by parts, one may use

$$\theta = \int_0^{\tau}\theta_{sc}(\tau-\tau')\frac{\partial}{\partial\tau'}H(\eta,\tau')\,\mathrm{d}\tau'. \quad (35)$$

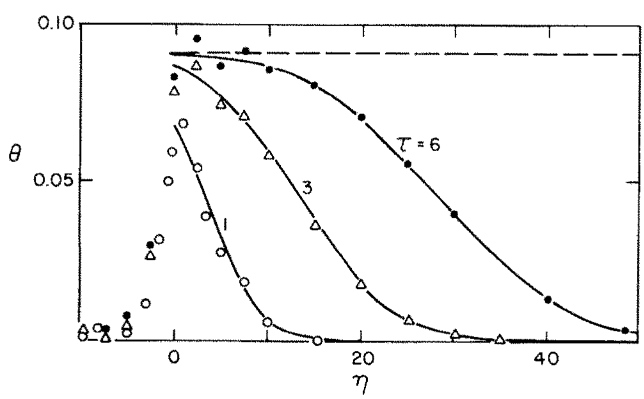

FIG. 4. Dependent variable profiles over space, at successive times, for a particular test case. Point values are numerical, solid lines analytical solutions. The dashed line shows the steady-state.

The last of these alternatives is simplest in general form, especially if θ_{sc} is a relatively simple function. The derivative in the integrand also eliminates error functions there, leaving exponentials and powers of τ. At the same time, the arguments of the exponentials are complicated, and in any given case integration of their products with $\theta_{sc}(\tau-\acute{\tau})$ and various fractional powers of τ may be quite unappetizing. In what follows, the evident feasibility of evaluating the integral in (34) recommended its use instead.

6. SINUSOIDAL SOURCE INTENSITY

The obvious particular $\theta_{sc}(\tau)$ to pursue is a sinusoid of arbitrary amplitude and frequency. Among other things, from that result, combined with the result for a constant θ_{sc}, one may construct the solution for an arbitrary $\theta_{sc}(\tau)$ by using its Fourier series. To this end, we consider

$$\theta_{sc}(\tau) = A \sin \omega\tau. \qquad (36)$$

In what follows, it is convenient to write

$$\frac{\partial}{\partial\acute{\tau}}\theta_{sc}(\tau-\acute{\tau}) = -Re\,\omega A \exp[i\omega(\tau-\acute{\tau})], \qquad (37)$$

so that from (34),

$$\theta = Re\,\omega A \int_0^\tau H(\eta,\acute{\tau}) \exp[i\omega(\tau-\acute{\tau})]\,d\acute{\tau}. \qquad (38)$$

Using the formula in Appendix B, one may evaluate (38) as

$$\begin{aligned}\theta = {} & k_1 \exp[(q^2-1)\tau/4 + (q+1)\eta/2] \\ & \times \operatorname{erfc}[(q/2)\sqrt{\tau} + (\eta/2)/\sqrt{\tau}] \\ & + Re\,k_2 \exp[i\omega\tau + (1+2b)\eta/2] \operatorname{erfc}[b\sqrt{\tau} + (\eta/2)/\sqrt{\tau}] \\ & + Re\,k_3 \exp[i\omega\tau + (1-2b)\eta/2] \operatorname{erfc}[(\eta/2)/\sqrt{\tau} - b\sqrt{\tau}],\end{aligned} \qquad (39)$$

where the constants k_1, k_2 and k_3 are given in Appendix B and

$$b^2 = \frac{1}{4} + i\omega. \qquad (40)$$

At any finite η, most of the terms in (39) drop away as time increases, leaving

$$\left.\begin{aligned}&\theta = Re\,2k_3 \exp[i\omega\tau + (1-2b)\eta/2],\\ &\tau \to \text{large}.\end{aligned}\right\} \qquad (41)$$

From (40) one may show that the real part of b is greater than one half. Thus (41) represents a long time solution for θ, which is periodic in space and time, and decays in amplitude downstream.

SUMMARY

Repeated application of integral transforms and solution of an integral equation yield a general solution for 1-dim. transport from a transfer dependent, spatially concentrated source. Using the solution to the case of constant source intensity, one achieves a general expression for the solution when the source intensity varies arbitrarily in time. Explicit expressions are derived for the solution of the fundamental case when the source intensity varies sinusoidally. In both this and the constant source case, quite simple expressions result for the long time solution. From the solution to the sinusoidally forced case, one may construct a solution for any specifiable time history of source intensity, using its Fourier series.

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APPENDIX A

In this Appendix, equation (25) is subjected to various manipulations and a Laplace transform, leaving an equation in the transformed dependent variable which is devoid of any differential or integral operators. This means ultimately that a double (Fourier–Laplace) transform must be inverted to obtain the final expressions for ϕ and hence θ. The particulars of the inversion process are also displayed below.

A1. *Additional transformations*

Laplace transformation of equation (25) with respect to τ produces an equation without the integral over τ. To this end one may rewrite (25) as

$$\bar{\phi}(\beta,\tau) = -q\int_0^\tau \exp[-\beta^2(\tau-\hat{\tau})]\,h(\hat{\tau})\mathrm{d}\hat{\tau} + f(\tau), \quad \text{(A.1)}$$

in which

$$h(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\phi}(\beta,\tau)\,\mathrm{d}\beta \quad \text{(A.2)}$$

and

$$f(\tau) \equiv \frac{q\theta_{sc}}{\frac{1}{4}+\beta^2}[\exp(\tau/4) - \exp(-\beta^2\tau)]\cdot \quad \text{(A.3)}$$

The Laplace transform with respect to τ of any function $g(\tau)$ is defined as

$$L_\tau\{g\} \equiv \int_0^\infty g(\hat{\tau})\exp(-s\hat{\tau})\,\mathrm{d}\hat{\tau}. \quad \text{(A.4)}$$

Using the convolution rule on (A.1) [23, p. 323, No. 7] yields an expression for Π, the Laplace transform of $\bar{\phi}$

$$L_\tau\{\bar{\phi}\} \equiv \Pi(\beta, s) = -qL_\tau\{\exp(-\beta^2\tau)\}\,L_\tau\{h(\tau)\} + L_\tau\{f\}. \quad \text{(A.5)}$$

Using the definition of $h(\tau)$ (A.2), one may express its Laplace transform in terms of Π, in integral form:

$$\begin{aligned} L_\tau\{h\} &= \int_{\hat{\tau}=0}^{\infty}\left\{\frac{1}{2\pi}\int_{\beta=-\infty}^{\infty}\bar{\phi}(\beta,\hat{\tau})\,\mathrm{d}\beta\right\}\exp(-s\hat{\tau})\,\mathrm{d}\hat{\tau} \\ &= \frac{1}{2\pi}\int_{\beta=-\infty}^{\infty}\left\{\int_{\hat{\tau}=0}^{\infty}\bar{\phi}(\beta,\hat{\tau})\exp(-s\hat{\tau})\mathrm{d}\hat{\tau}\right\}\mathrm{d}\beta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Pi(\beta,s)\,\mathrm{d}\beta. \end{aligned} \quad \text{(A.6)}$$

Substitution of (A.6) into (A.5), and the direct evaluation of $L_\tau\{\exp(-\beta^2\tau)\}$ and $L_\tau\{f\}$ through the definition (A.4) leads to an integral equation in $\Pi(\beta, s)$ containing only a single integral:

$$\Pi(\beta,s) = \frac{-q}{s+\beta^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Pi(\beta,s)\mathrm{d}\beta + \frac{q\theta_{sc}}{\beta^2+\frac{1}{4}}\left[\frac{1}{s-\frac{1}{4}} - \frac{1}{s+\beta^2}\right]. \quad \text{(A.7)}$$

From (A.7) one may obtain an equation in $\Pi(\beta, s)$ devoid of all integral–differential operators by using a method suggested in general form by Hildebrand [24, p. 246 ff.], and applied as below.

To begin, denote the integral in (A.7) as B, that is,

$$B(s) \equiv \int_{-\infty}^{\infty}\Pi(\beta,s)\mathrm{d}\beta \quad \text{(A.8)}$$

so that (A.7) may be rewritten as

$$\Pi(\beta,s) = \frac{-q}{2\pi}\frac{1}{s+\beta^2}B(s) + \frac{q\theta_{sc}}{\beta^2+\frac{1}{4}}\left[\frac{1}{s-\frac{1}{4}} - \frac{1}{s+\beta^2}\right]. \quad \text{(A.9)}$$

Now one may solve for $B(s)$, using the definition (A.8), by integrating (A.9) with respect to β. When the result is substituted back into (A.9), the desired solution for $\Pi(\beta, s)$ is obtained. Equation (A.9) becomes

$$B(s) = \frac{-qB(s)}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\beta}{(s+\beta^2)} + \frac{q\theta_{sc}}{\left(s-\frac{1}{4}\right)}\int_{-\infty}^{\infty}\frac{\mathrm{d}\beta}{\left(\beta^2+\frac{1}{4}\right)} - q\theta_{sc}\int_{-\infty}^{\infty}\frac{\mathrm{d}\beta}{\left(\beta^2+\frac{1}{4}\right)(s+\beta^2)}. \quad \text{(A.10)}$$

Using available evaluations of the integrals in (A.10) [25, p. 77], [26, p. 289, formula 3.223.1] one obtains

$$B(s) = \frac{q\theta_{sc}\pi}{\left(s-\frac{1}{4}\right)(q/2+\sqrt{s})}. \quad \text{(A.11)}$$

Substituting (A.11) into (A.9) provides the result

$$\Pi(\beta,s) \equiv L_\tau\{\bar{\phi}\} = \frac{-(q/2)}{(s+\beta^2)}\frac{q\theta_{sc}}{\left(s-\frac{1}{4}\right)(q/2+\sqrt{s})}$$

$$+\frac{q\theta_{sc}}{\left(\beta^2+\frac{1}{4}\right)\left(s-\frac{1}{4}\right)}-\frac{q\theta_{sc}}{\left(\beta^2+\frac{1}{4}\right)(s+\beta^2)} \quad \text{(A.12a)}$$

or

$$\Pi(\beta,s)=L_\tau\{\bar{\phi}_1\}+L_\tau\{\bar{\phi}_2\}+L_\tau\{\bar{\phi}_3\}, \quad \text{(A.12b)}$$

where the terms on the right hand sides of (A.12a) and (A.12b) correspond respectively, and

$$\bar{\phi}=\bar{\phi}_1+\bar{\phi}_2+\bar{\phi}_3. \quad \text{(A.13)}$$

The variable $\Pi(\beta, s)$ is completely and explicitly specified by equation (A.12). It represents the Laplace transform of the Fourier transform of ϕ. When those transforms are inverted, as below, solution for θ may be expressed explicitly, via (14).

A.2. *Inverse transformation*

By virtue of the linearity of the back-transformation operations, the inverse transforms may be applied term by term to equation (A.12). Also, either inversion may be performed first on each term, according to convenience. For the first term on the right-hand side of (A.12), it simplifies matters if one inverts the Fourier transform first:

$$L_\tau\{\phi_1\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(-q/2)\,q\theta_{sc}\exp(-i\beta\eta)}{(s+\beta^2)\left(s-\frac{1}{4}\right)(q/2+\sqrt{s})}\,d\beta$$

$$=\frac{-q^2\theta_{sc}\exp(-\eta\sqrt{s})}{4\left(s-\frac{1}{4}\right)\sqrt{s}\,(q/2+\sqrt{s})}\cdot \quad \text{(A.14)}$$

For evaluating the integral in (A.14) [26, p. 406, formula 3.723.2] it must be assumed that $\eta>0$. Thus the solution to be obtained is applicable downstream if V is taken as positive, and upstream if a negative V is used.

The Laplace transform remaining in (A.14) may be inverted as [23, formula 87, p. 329]

$$\phi_1(\eta,\tau)=\left(\frac{-q^2\theta_{sc}}{4}\right)\exp[(q/2)\eta+\tau/4]\int_0^\tau\exp(q^2-1)\,\dot{\tau}/4$$

$$\times\,\mathrm{erfc}\left[(q/2)\sqrt{\dot{\tau}}+\frac{(\eta/2)}{\sqrt{\dot{\tau}}}\,d\dot{\tau}\right]. \quad \text{(A.15)}$$

When the integral in (A.15) with respect to $\dot{\tau}$ is evaluated [27, 2.6.4] the result is

$$\phi_1(\eta,\tau)=q^2\theta_{sc}\left\{\frac{\exp(q\eta/2+q^2\tau/4)}{1-q^2}\,\mathrm{erfc}\,[(q/2)\sqrt{\tau}+(\eta/2)/\sqrt{\tau}]\right.$$

$$+\frac{\exp(\eta/2+\tau/4)}{2(q-1)}\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}+(\eta/2)/\sqrt{\tau}\right]+\frac{\exp(-\eta/2+\tau/4)}{2(q+1)}$$

$$\left.\times\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}-(\eta/2)/\sqrt{\tau}\right]\right\}-\frac{q^2\theta_{sc}\exp(\tau/4-\eta/2)}{q+1},\quad q\neq 1. \quad \text{(A.16)}$$

To treat the second term on the right hand side in (A.12), it is best to invert the Laplace transform first. The result is [23, p. 324]

$$\bar{\phi}_2=\frac{q\theta_{sc}\exp(\tau/4)}{\beta^2+\frac{1}{4}}. \quad \text{(A.17)}$$

One may inverse-Fourier transform (A.17) [26, p. 406, formula 3.723.2] to obtain

$$\phi_2=q\theta_{sc}\exp[(\tau/4)-(\eta/2)] \quad \text{(A.18)}$$

The last term on the right-hand side of (A.12) may be inverse-Laplace transformed to provide [23, p. 324]

$$\bar{\phi}_3=\frac{-q\theta_{sc}\exp(-\beta^2\tau)}{\beta^2+\frac{1}{4}}. \quad \text{(A.19)}$$

Equation (A.19) may be inverse-Fourier transformed [27, 3.15.1] as

$$\phi_3=\left(\frac{q\theta_{sc}}{2}\right)\exp(\tau/4)\left\{\exp(\eta/2)\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}+(\eta/2)/\sqrt{\tau}\right]\right.$$

$$\left.+\exp(-\eta/2)\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}-(\eta/2)/\sqrt{\tau}\right]\right\}\cdot \quad \text{(A.20)}$$

The dependent variable ϕ is given by the sum of equations (A.16), (A.18) and (A.20), that is

$$\phi=\phi_1+\phi_2+\phi_3. \quad \text{(A.21)}$$

When one performs this summation, using (14) to express things in terms of θ, and employs additional straightforward rearrangements and simplifications of expressions, the result is

$$\theta(\eta,\tau)=\left\{\frac{q^2\theta_{sc}}{1-q^2}\exp[(q+1)\eta/2+(q^2-1)\tau/4]\right\}$$

$$\times\ \mathrm{erfc}[(q/2)\sqrt{\tau}+(\eta/2)/\sqrt{\tau}]$$

$$+\frac{q\theta_{sc}\exp^2\eta}{2(q-1)}\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}+(\eta/2)/\sqrt{\tau}\right]$$

$$-\frac{q\theta_{sc}}{2(q+1)}\,\mathrm{erfc}\left[\frac{1}{2}\sqrt{\tau}-(\eta/2)/\sqrt{\tau}\right]+\frac{q\theta_{sc}}{q+1}. \quad \text{(A.22)}$$

APPENDIX B

All the integrals in (38) may be evaluated using the formula [27, 2.6.4]:

$$\int\exp[(a^2-b^2)x]\,\mathrm{erfc}[a\sqrt{x}+c\sqrt{x}]\,dx$$

$$=\frac{\exp(a^2-b^2)x}{a^2-b^2}\,\mathrm{erfc}[a\sqrt{x}+c/\sqrt{x}]$$

$$-\frac{\exp[-2(a-b)c]}{2(a^2-b^2)}(1+a/b)\,\mathrm{erfc}[b\sqrt{x}+c/\sqrt{x}]$$

$$+\frac{\exp[-2(a+b)c]}{2(a^2-b^2)}(1-a/b)\,\mathrm{erfc}[b\sqrt{x}-c/\sqrt{x}]+C_1, \quad \text{(B.1)}$$

where C_1 is a constant of integration. In each integral b^2 corresponds to the complex quantity $1/4+i\omega$.

When the integrations are carried out, yielding the result in (39), the constants which appear are

$$k_1=\frac{-q^2\omega A}{4[(q^2-1)^2/16+\omega^2]}, \quad \text{(B.2)}$$

$$k_2=\frac{q^2A\omega(1+q/2b)}{2(q^2-1)(q^2/4-b^2)}+\frac{iAq}{4}\left[\frac{1+1/2b}{1-q}-\frac{1-1/2b}{q+1}\right], \quad \text{(B.3)}$$

$$k_3=\frac{q^2A\omega(1-q/2b)}{2(q^2-1)(q^2/4-b^2)}-\frac{iqA(1+1/2b)}{4(q+1)}+\frac{iqA(1-1/2b)}{4(1-q)} \quad \text{(B.4)}$$

TRANSFERT UNIDIRECTIONNEL A PARTIR D'UNE SOURCE THERMIQUE TRES CONCENTREE

Résumé—On considère les situations de transfert de chaleur et de masse pour lesquelles l'intensité d'une source ou d'un puits peut-être exprimée en fonction d'un débit inconnu. Ceci se rencontre quand le transfert entre la source et le milieu est gouverné par une valeur différentielle d'une variable indépendante qui est inconnue parce que la valeur de réponse du milieu est inconnue. Des complexités mathématiques apparaissent quand en plus la source est spatialement très concentrée, par rapport à la dimension du domaine. Une équation de convection unidirectionnelle valable pour ce cas peut être résolue par l'usage simultané de la transformation de Fourier et de son inverse dans la même équation, en même temps qu'avec une autre transformation et une autre manipulation. A partir de la solution obtenue dans le cas d'une intensité constante de source, on peut construire une expression générale de la solution lorsque l'intensité varie avec le temps. Une solution explicite est obtenue pour le cas fondamental d'une intensité de source fonction sinusoïdale du temps.

EINDIMENSIONALER TRANSPORT VON EINER HOCHKONZENTRIERTEN WÄRME- BZW. STRÖMUNGSQUELLE

Zusammenfassung — Sowohl bei der Wärmeübertragung als auch beim Stoffaustausch treten Situationen auf, in denen eine als Quelle oder Senke betrachtete Größe nur durch eine unbekannte Ergiebigkeit ausgedrückt werden kann. Dieses ist der Fall, wenn die den Transportvorgang zwischen der Quelle und dem Medium bestimmende Differenz einer abhängigen Variablen unbekannt ist, weil der zugehörige Wert des Mediums nicht bekannt ist. Es ergeben sich weitere mathematische Komplizierungen, wenn die Quelle zusätzlich noch stark konzentriert ist im Verhältnis zur Größe des Gesamtbereiches. Eine eindimensionale Konvektions-Diffusions-Transportgleichung, die diesem Fall entspricht, kann durch gleichzeitige Anwendung der Fourier-Transformation und ihrer Inversen in derselben Gleichung und durch weitere Transformation und Manipulation gelöst werden. Aus der Lösung, die man für den Fall konstanter Quellenergiebigkeit erhält, kann man einen allgemeinen Lösungsausdruck für zeitlich beliebig veränderliche Quellenergiebigkeit aufbauen. Man erhält explizite Ausdrücke für die Lösung des Fundamentalfalles zeitlich sinusförmig veränderlicher Quellenergiebigkeit.

ОДНОМЕРНЫЙ ПЕРЕНОС ОТ СИЛЬНО СКОНЦЕНТРИРОВАННОГО ИСТОЧНИКА

Аннотация — В случае как тепло-, так и массопереноса возникают ситуации, когда источник или сток характеризуется мощностью, которая может быть выражена через искомую скорость переноса от источника в поле потока. Это происходит в том случае, когда перенос между источником и средой возникает под действием разности значений зависимой переменной, являющейся неизвестной величиной в реагирующей среде. Математическое описание особенно затруднено в том случае, когда в дополнение ко всему источник является сильно сконцентрированным в пространстве (по отношению к размерам рассматриваемой области). Пригодное для этого случая одномерное уравнение, описывающее конвективный и диффузионный перенос, может быть решено путем использования преобразования Фурье или другого преобразования. Из решения, полученного для источника постоянной мощности, можно вывести общее выражение для случая источника, мощность которого произвольно изменяется во времени. Получены выражения в явном виде для источника с мощностью, синусоидально зависящей от времени.